International Journal of Theoretical Physics, Vol. 37, No. 9, 1998

Multiple Soliton-Like Solutions for (2+1)-Dimensional Dispersive Long-Wave Equations

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Received June 18, 1998

By using a homogeneous balance method, multiple-solitonlike solutions of the (2+1)-dimensional dispersive long-wave equation are constructed. The method used here can be generalized to a wide class of nonlinear evolution equations.

Since soliton phenomena were first observed by Scott Russell in 1834 (Scott Russell, 1838) and the KdV equation was solved by the inverse scattering method by Gardner *et al.* (1967), the study of solitons and the related issue of the construction of solutions to a wide class of nonlinear equations has become one of the most exciting and extremely active areas of research. Various methods for obtaining soliton solutions of the nonlinear evolution equations have been proposed. Among these method are Hirota's method (Hirota, 1971), the Backlund transformation (Miura, 1978), the Darboux transformation (Gu and Zhou, 1987) the Riemann method (Belinsky and Zakharov (1978), Painlevé expansions (Cariello and Tabor, 1989), and several 'ansatz' methods (Wang, 1993; Malfleit, Lu *et al.*, 1993). Recently Wang introduced a homogeneous balance method (Wang, 1995, 1996a) and obtained the solitary wave solutions of some nonlinear evolutions. In this paper, we generalize the homogeneous balance method and give multiple-solitonlike solutions of the (2+1)-dimensional dispersive long-wave equations (Boiti *et al.*, 1987)

$$u_{ty} + \eta_{xx} + u_x u_y + u u_{xy} = 0 \tag{1}$$

$$\eta_t + (u\eta + u + y_{xy})_x = 0 \tag{2}$$

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In order to solve equations (1) and (2), we suppose that the solution is of the form

$$\eta(x, y, t) = f(w)_{xy} + a = f''w_x w_x + f'w_{xy} + a \tag{3}$$

$$u(x, y, t) = f(w)_x + b = f'w_x + b \tag{4}$$

where the functions $\eta(x, y, t)$ and $u(x, y, t)$ are expressed by a function $f(w)$ of one argument $w(x, y, t)$ only, whereas $f(w)$ and the constants a and b are to be determined later. We will see that using the anaẗsz (3) and (4), the nonlinear and dispersive effects in Equations (1) and (2) can be partially balanced. Therefore the only task in the following is to find the functions $f(w)$ and $w(x, y, t)$ as well as the constants a and b such that (3) and (4) actually satisfy (1) and (2).

From (3) and (4), we can easily deduce that

$$u_t = f''w_x w_t + f'w_{tx} \tag{5}$$

$$u_x = f''w_x^2 + f'w_{xx} \tag{6}$$

$$u_y = f''w_x w_y + f'w_{xy} \tag{7}$$

$$u_{ty} = f^{(3)}w_x w_y w_t + f''(w_{xy}w_t + w_x w_{ty} + w_y w_{tx}) + f'w_{txy} \tag{8}$$

$$u_{xy} = f^{(3)}w_x^2 w_y + f''(2w_x w_{xy} + w_y w_{xx}) + f'w_{xxy} \tag{9}$$

$$u_{xxy} = f^{(4)}w_x^3 w_y + f^{(3)}(3w_x w_y w_{xx} + 3w_x^2 w_{xy}) + f''(3w_{xx}w_{xy} + 3w_x w_{xxy} + w_y w_{xxx}) + f'w_{xxxy} \tag{10}$$

$$\eta_t = f^{(3)}w_x w_y w_t + f''(w_x w_{ty} + w_{tx}w_y + w_{xy}w_t) + f'w_{txy} \tag{11}$$

$$\eta_x = f^{(3)}w_x^2 w_y + f''(w_y w_{xx} + 2w_x w_{xy}) + f'w_{xxy} \tag{12}$$

$$\eta_{xx} = f^{(4)}w_x^2 w_y + f^{(3)}(3w_x^2 w_{xy} + 3w_x w_{xx} w_y) + f''(2w_x w_{xxy} + 3w_{xx}w_{xy} + w_y w_{xxx}) + f'w_{xxxy} \tag{13}$$

Substituting equations (5)–(13) into the left-hand side of equations (1) and (2), we obtain

$$(f^{(4)} + f'f^{(4)} + f''^2)w_x^2 w_y + f^{(3)}(bw_x^2 w_y + 3w_x w_{xx}x_y + 3w_x^2 w_{xy} + w_x w_y w_t) + f'f''(3w_x^2 w_{xy} + 2w_x w_y w_{xx}) + f''(w_{xy}w_t + w_x w_{ty} + w_y w_{tx} + 3w_{xx}w_{xy} + w_y w_{xxx} + 3w_x w_{xxy} + 2bw_x w_{xy} + bw_y w_{xx}) + f'^2(w_{xx}w_{xy} + w_x w_{xxy}) + f'(w_{txy} + w_{xxxy} + bw_{xxy}) = 0 \tag{14}$$

$$(f^{(4)} + f''^2 + f'f^{(3)})w_x^2 w_y + f^{(3)}\,(bw_{xwy}^2 + 3w_x w_{xx}x_y + 3w_x^2 w_{xy} + w_x w_y w_t) + f'f''(3w_x^2 w_{xy} + 2w_x w_y w_{xx}) + f''(w_{xy}w_t + w_x w_{ty} + w_y w_{tx} + 3w_x w_{xxy}$$

$$+ 3w_{xx}w_{xy} + w_y w_{xxx} + 2bw_x w_{xy} + bw_y w_{xx} + aw_x^2 + w_x^2)$$

$$+ f'^2(w_{xx}w_{xy} + w_x w_{xxy}) + f'(w_{txy} + w_{xxxy} + bw_{xxy} + aw_{xx} + w_{xx}) \quad (15)$$

The expressions (14) and (15) indicates that the nonlinear terms and the highest order partial derivative terms in (1) and (2) have been partially balanced. That is why we assume that the solution of (1) and (2) is of the form of (3) and (4).

To simplify expressions (14) and (15), we further suppose that

$$f^{(4)} = -(f'f^{(3)} + f''^2) \quad (16)$$

and thus we have

$$f'' = -\frac{1}{2}f'^2 \quad (17)$$

$$f'f'' = -f^{(3)} \quad (18)$$

Making use of equations (16) and (18), we can simplify expressions (14) and (15)

$$f^{(3)}(bw_x^2 w_y + w_x w_{xx} w_y + w_x w_y w_t) + f''(w_{xy}w_t + w_x w_{ty} + w_y w_{tx} + w_{xx}w_{xy} + w_x w_{xxy} + w_y w_{xxx} + bw_y w_{xx} + 2bw_x w_{xy}) + f'(w_{txy} + w_{xxxy} + bw_{xxy}) = 0 \quad (19)$$

$$f^{(4)}(bw_x^2 w_y + w_x w_{xx} w_y + w_x w_y w_t) + f''(w_{xy}w_t + w_x w_{ty} + w_y w_{tx} + w_x w_{xxy} + w_{xx}w_{xy} + w_y w_{xxx} + 2bw_x w_{xy} + bw_x w_{xx} + aw_x^2 + w_x^2) + f'(w_{txy} + w_{xxxy} + bw_{xxy} + aw_{xx} + w_{xx}) = 0 \quad (20)$$

Setting the coefficients of $f^{(4)}$, f'', and f' in equations (19) and (20) to zero yields a set of equations for $w(x, y, t)$:

$$bw_x^2 w_y + w_x w_{xx} w_y + w_x w_y w_t = 0 \quad (21)$$

$$w_{xy}w_t + w_x w_{ty} + w_y w_{xt} + w_{xx}w_{xy} + w_x w_{xxy} + w_y w_{xxx} + bw_y w_{xx} + 2bw_x w_{xy} = 0 \quad (22)$$

$$w_{txy} + w_{xxxy} + bw_{xxy} = 0 \quad (23)$$

$$w_{xy}w_t + w_x w_{ty} + w_y w_{xy} + w_x w_{xxy} + w_{xx}w_{xy} + w_y w_{xxx} + 2bw_x w_{xy} + bw_y w_{xx} + aw_x^2 + w_x^2 = 0 \quad (24)$$

$$w_{xyt} + w_{xxxy} + bw_{xxy} + aw_{xx} + w_{xx} = 0 \quad (25)$$

Provided that

$$a = -1 \tag{26}$$

we have

$$w_x w_y (bw_x + w_{xx} + w_t) = 0 \tag{27}$$

$$(bw_x + w_{xx} + w_t)_{xy} = 0 \tag{28}$$

$$w_{xy}(bw_x + w_{xx} + w_t) + w_y(bw_x + w_{xx} + w_t)_x + w_x(bw_x + w_{xx} + w_t)_y = 0 \tag{29}$$

From equations (27)–(29) we can get a special solution which satisfies equation, (27)–(29) as follows:

$$w(x, y, t) = + \sum_{j=1}^{n} \exp[k_j(y)x + l_j(y) - (bk_j(y) + k_j^2(y))t] \tag{30}$$

where $k_j(y)$ and $l_j(y)$ are two arbitary functions of y.

Solving (17) yields

$$f = 2 \ln w \tag{31}$$

Now substituting equations (26), (30), and (31) into Eequations (3) and (4), we obtain the multiple-solitonlike solutions of (1) and (2):

$$u = \frac{2\sum_{j=1}^{n} k_j(y) \exp[k_j(y)x + l_j(y) - (bk_j(y) + k_j^2(y))t]}{1 + \sum_{j=1}^{n} \exp[k_j(y)x + l_j(y) - (bk_j(y) + k_j^2(y))t]} + b \tag{32}$$

$$\begin{aligned}
\eta = {} & \{-2 \sum_{j=1}^{n} \sum_{j=1}^{n} k_j(y) l_i(y) \exp[k_j(y)x + l_j(y) - (bk_j(y) + k_j^2(y))t] \\
& \times \exp[k_j(y)x + l_j(y) - (bk_j(y) + k_j^2(y))t]\} \\
& \times \{1 + \sum_{j=1}^{n} \exp[k_j(y)x + l_j(y) - (bk_j(y) + k_j^2(y))t]\}^{-1} \\
& + \{2 + \sum_{j=1}^{n} k_j(y) l_j(y) \exp[k_j(y)x + l_j(y) - (bk_j(y) + k_j^2(y))t]\} \\
& \times \{1 + \sum_{j=1}^{n} \exp[k_j(y)x + l_j(y) - (bk_j(y) + k_j^2(y))t]\}^{-1}
\end{aligned} \tag{33}$$

To understanding the meaning of solutions (32) and (33), we discuss some special cases:

(i) For $n = 1$, we obtain

$$u = k_1(y)\tanh\left[\frac{1}{2}(k_1(y)x + l_1(y) - (bk_1(y) + k_1^2(y))t)\right] + b + 1 \quad (34)$$

$$\eta = \frac{1}{2}k_1(y)l_1(y)\,\mathrm{sech}^2\left[\frac{1}{2}(k_1(y)x + l_1(y) - (bk_1(y) + k_1^2(y))t\right] - (B5)$$

Obviously, u is a bell-shaped soliton solution, and η is a kinklike soliton solution.

(ii) For $n = 2$, we can construct the two-soliton solution as follows:

$$y = \frac{2k_1(y)\exp(\xi_1) + 2k_2(y)\exp(\xi_2)}{1 + \exp(\xi_1) + \exp(\xi_2)} + b \quad (36)$$

$$\begin{aligned}\eta = {}& [2k_1(y)l_1(y)\exp(\xi_1) + 2k_2(y)l_2(y)\exp(\xi_2)\\ &+ 2(l_1(y) - l_2(y))(k_1(y) - k_2(y))\exp(\xi_1 + \xi_2)]\\ &\times [1 + \exp(\xi_2) + \exp(\xi_2)]^{-2} - 1\end{aligned} \quad (37)$$

where

$$\xi_1 = k_1(y)x - (bk_1(y) + k_1^2(y))t \quad (38)$$

$$\xi_2 = k_2(y)x - (bk_2(y) + k_2^2(y))t \quad (39)$$

(iii) For $n = 3$, we can construct the three-soliton solution as follows:

$$\begin{aligned}u = {}& [2k_1(y)\exp(\xi_1) + 2k_2(y)\exp(\xi_2) + 2k_3(y)\exp(\xi_3)]\\ &\times [1 + \exp(\xi_1) + \exp(\xi_2) + \exp(\xi_3)]^{-1} + b\end{aligned} \quad (40)$$

$$\begin{aligned}\eta = {}& [2k_1(y)l_1(y)\exp(\xi_1) + 2k_2(y)l_2(y)\exp(\xi_2) + 2k_3(y)l_3(y)\exp(\xi_3)\\ &+ 2(l_1(y) - l_2(y))k_1(y) - k_2(y))\exp(\xi_1 + \xi_2)\\ &+ 2(l_1(y) - l_3(y))(k_1(y) - k_3(y))\exp(\xi_1 + \xi_3)\\ &+ 2(l_2(y) - l_3(y))(k_2(y) - k_3(y))\exp(\xi_2 + \xi_3)]\\ &\times [1 + \exp(\xi_1) + \exp(\xi_2) + \exp(\xi_3)]^{-2} - 1\end{aligned} \quad (41)$$

where

$$\xi_3 = k_3(y)x - bk_3(y) + k_3^2(y))t \quad (42)$$

Moreover, we can construct other kinds of multiple-solitonlike solutions as follows:

(iv) For $n = 2$,

$$u = \frac{2k_1(y)[\exp(\zeta_1) + K\exp(\zeta_1 + \zeta_2)]}{1 + \exp(\zeta_1) + \exp(\zeta_2) + K\exp(\zeta_1 + \zeta_2)} + b \tag{43}$$

$$\eta = \frac{2(K - 1)k_1(y)l_2(y)\exp(\zeta_1 + \zeta_2)}{[1 + \exp(\zeta_1) + \exp(\zeta_2) + K\exp(\zeta_1 + \zeta_2)]^2} - 1 \tag{44}$$

where

$$\zeta_1 = k_1(y)x - (bk_1(y) + k_1^2(y))t \tag{45}$$

$$\zeta_2 = l_2 y \tag{46}$$

(v) For $n = 3$

$$u = \frac{2[k_1(y)e\zeta_1 + k_3(y)e\zeta_3 + Kk_1(y)e\zeta_1 + \zeta_2 + Kk_3(y)e\zeta_2 + \zeta_3}{1 + e\zeta_1 + e\zeta_2 + e\zeta_3 + K\zeta_1 + \zeta_2 + K\zeta_2 + \zeta_3} + b \tag{47}$$

$$u = \frac{2(k - 1)[k_1(y)l_2(y)e\zeta_1 + \zeta_2 + k_3(y)l_2(y)e\zeta_2 + \zeta_3]}{[1 + e\zeta_1 + e\zeta_2 + e\zeta_3 + Ke\zeta_1 + \zeta_2 + Ke\zeta_2 + \zeta_3]^2} + 1 \tag{48}$$

where

$$\zeta_3 = k_3(y)x - (bk_3(y) + k_3^2(y))t \tag{49}$$

In summary, the multiple-solitonlike solution of the (2+1)-dimensional dispersive long-wave equation can be obtained by using the homogeneous balance method. The method used here, which is very coincise and basic, can be conjecturally generalized to deal with other nonlinear evolution equations. We would like to study this in future work.

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